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ON PERIODIC ALMOST DOUBLY ASYMPTOTIC SOLUTIONS OF THE BOUNDED CIRCULAR THREE-BODY PROBLEM*

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In the course of investigating by numerical integration methods the solutions of the bounded, circular three-body problem doubly asymptotic with respect to the rectilinear libration points $L_{1,2,3}$, it was found that a periodic solution always exists near the doubly asymptotic solution, and a problem of establishing an analytic relationship between such solutions was therefore formulated (**).

Below, a theorem is proved from which it follows that a periodic solution exists in any neighborhood of a solution of the bounded, circular three-body problem, doubly asymptotic with respect to the rectilinear libration point.

Let U be a region in a four-dimensional space R^4 with coordinates x_1, x_2, y_1, y_2. We consider in U a Hamiltonian system

$$x_\nu^{\cdot} = H_{y_\nu}, \quad y_\nu^{\cdot} = -H_{x_\nu} \quad (\nu = 1, 2) \tag{1}$$

with real, analytic Hamiltonian function $H(z)$, $z = (x_1, x_2, y_1, y_2)$. We assume that:

system (1) has a fixed point z_0 (i.e. $\operatorname{grad} H(z_0) = 0$);

the matrix of a system linearized about the point z_0 has eigenvalues $\alpha_1, -\alpha_1, \alpha_2 i, -\alpha_2 i$ where $\alpha_{1,2} > 0$, $i = \sqrt{-1}$; system (1) has a doubly asymptotic solution $z^*(t)$: $\lim z^*(t) = z_0$ as $t \to \pm\infty$.

Theorem. A periodic solution of the system (1) exists in the phase space U in any neighborhood of the solution $z^*(t)$.

Proof. According to the Moser theorem (see /1,2/) a real analytic canonical change of variables

$$z = \Phi(\zeta), \quad \zeta = (\xi_1, \xi_2, \eta_1, \eta_2), \quad \Phi(0) = z_0$$

exists in a sufficiently small neighborhood of the point z_0, which reduces (1) to the normal form

$$\xi_\nu^{\cdot} = F_{\eta_\nu}, \quad \eta_\nu^{\cdot} = -F_{\xi_\nu} \quad (\nu = 1, 2) \tag{2}$$

$$F(\zeta) = H(\Phi(\zeta)) - H(z_0) = F(\omega_1, \omega_2) = \alpha_1\omega_1 + \alpha_2\omega_2 + \ldots, \qquad \omega_1 = \xi_1\eta_1, \quad \omega_2 = {}^1/_2(\xi_2^2 + \eta_2^2)$$

The system (2) is integrable, and we have

$$\omega_\nu = \omega_\nu^\circ = \text{const} \ (\nu = 1, 2), \qquad \xi_1 = \xi_1^\circ \exp[a_1(\omega^\circ)t], \quad \xi_2 = \xi_2^\circ \cos a_2(\omega^\circ)t + \eta_2^\circ \sin a_2(\omega^\circ)t \tag{3}$$

$$\eta_1 = \eta_1^\circ \exp[-a_1(\omega^\circ)t], \quad \eta_2 = -\xi_2^\circ \sin a_2(\omega^\circ)t + \eta_2^\circ \cos a_2(\omega^\circ)t, \qquad a_\nu(\omega^\circ) = F_{\omega_\nu}(\omega^\circ), \quad \omega^\circ = (\omega_1^\circ, \omega_2^\circ)$$

The existence of a doubly asymptotic solution $z^*(t)$ means that a certain point $z_1 = \Phi(\zeta_1)$, $\zeta_1 = (\xi_1^*, 0, 0, 0)$, $\xi_1^* \neq 0$ will pass, under the action of the phase flux, to a point $z_2 = \Phi(\zeta_2)$, $\zeta_2 = (0, \eta_1^*, 0, 0)$, $\eta_1^* \neq 0$. Here $|\xi_1^*|, |\eta_1^*|$ can be taken arbitrarily small.

Let $|\xi_1^*| \neq 0$ be sufficiently small. Then $F_{\eta_1}(\zeta_1) = \alpha_1\xi_1^* + O(\xi_1^{*2}) \neq 0$ and using the theorem on the implicit function near the point z_1 we can pass from the coordinates ζ to the coordinates $\xi_1, \xi_2, \eta_2, \chi = F(\zeta) = H(z) - H(z_0)$. Similarly, for sufficiently small $|\eta_1^*| \neq 0$ we can pass, near the point z_2, from the coordinates ζ to $\xi_2, \eta_1, \eta_2, \chi$.

Let us consider the mapping φ_h^1 of the two-dimensional area $S_h^1: \{\xi_1 = \xi_1^*, \chi = h\}$ with coordinates ξ_2, η_2 onto the two-dimensional area $S_h^2: \{\eta_1 = \eta_1^*, \chi = h\}$ with coordinates ξ_2, η_2 under the action of the phase flux. We denote $(\xi_2, \eta_2) = x$. Using the standard arguments we can show that the mapping $\varphi_h^1(x)$ is analytic in h, x when $|h|, |x|; \varphi_0^1(0) = 0$ are sufficiently small. On the other hand, in the circle $D_h: \{\omega_2 < c|h|\}$ where $c > 0$ is sufficiently small and $|h| \neq 0$ are also sufficiently small and such that $\operatorname{sign} h = \varkappa = \operatorname{sign}(\xi_1^*\eta_1^*)$, the mapping $\varphi_h^2: D_h \to S_h^1$ is defined according to the formulas (3) by

$$\varphi_h^2 \left\| \begin{matrix} \xi_2 \\ \eta_2 \end{matrix} \right\| = \left\| \begin{matrix} \xi_2 \cos\theta(h, \omega_2) + \eta_2 \sin\theta(h, \omega_2) \\ -\xi_2 \sin\theta(h, \omega_2) + \eta_2 \cos\theta(h, \omega_2) \end{matrix} \right\|, \qquad \theta(h, \omega_2) = \frac{a_2(\omega_1(h, \omega_2), \omega_2)}{a_1(\omega_1(h, \omega_2), \omega_2)} \ln \frac{\xi_1^*\eta_1^*}{\omega_1(h, \omega_2)}$$

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**) See Lidov, M. L. and Vashkov'iak, M. A. Doubly asymptotic symmetric orbits in the plane, bounded circular three-body problem. Moscow, Preprint In-ta prikl.matem. Akad. Nauk SSSR, No.15,1975. The solution is called doubly asymptotic with respect to a fixed point if it tends to this point as $t \to \pm\infty$.

where $\omega_1(h, \omega_2) = h / \alpha_1 + O(h^2 + \omega_2)$ is a solution of the equation $F(\omega_1, \omega_2) = h$ for ω_1, for small $|\omega_1|, |h|, |\omega_2|$, analytic in the neighborhood of $h = \omega_2 = 0$.

Thus for sufficiently small $|h| \neq 0, \varkappa h > 0$ we obtain the mapping $\varphi_h = \varphi_h^1 \varphi_h^2 : D_h \to D_h$. The fixed points

$$\varphi_h(x) = \varphi_h^1 \varphi_h^2(x) = x \tag{4}$$

of this mapping have the corresponding periodic solutions of the system (1). To prove the theorem it is sufficient to construct a sequence $\{h_n\} \to 0$ such, that the corresponding sequence of solutions of (4) also $\{x_n = x(h_n)\} \to 0$.

Instead of (4), we shall now consider the equivalent equation

$$\varphi_h^2(x) = (\varphi_h^1)^{-1}(x) \tag{5}$$

Substitution $x = \varepsilon \bar{x}, h = \varkappa \varepsilon^2$ reduces the equation (5) to the form

$$\Psi(\bar{x}, \varepsilon) = f(\bar{x}, \varepsilon, \theta(\varepsilon), \mu(\varepsilon)) = 0, \quad \theta(\varepsilon) = \theta(\varkappa\varepsilon^2, 0) = \frac{\alpha_2}{\alpha_1} \ln \frac{|\xi_1^* \eta_1^*| \alpha_1}{\varepsilon^2} \to \infty, \quad \varepsilon \to 0, \quad \mu(\varepsilon) = \varepsilon^2 \theta(\varepsilon) \to 0, \quad \varepsilon \to 0 \tag{6}$$

where the function $f(\bar{x}, \varepsilon, \theta, \mu)$ is analytic in $\bar{x}, \varepsilon, \theta$ and μ for sufficiently small $|\bar{x}|, |\varepsilon|, |\mu|$ and 2π-periodic in θ

$$f(0, 0, \theta, 0) \equiv 0, \quad \frac{\partial f}{\partial \bar{x}}(0, 0, \theta, 0) = \begin{Vmatrix} \cos\theta - a_{11} & \sin\theta - a_{12} \\ -\sin\theta - a_{21} & \cos\theta - a_{22} \end{Vmatrix}$$

where $(a_{ij}) = A$ is the matrix of the linear part of the mapping $(\varphi_0^1)^{-1}$ at the point $x = 0$. When $\theta(\varepsilon) \pmod{2\pi} = \theta^*$ is fixed, we can write the equation (6) in the form of a system

$$f(\bar{x}, \varepsilon, \theta^*, \mu) = 0, \quad \theta(\varepsilon) = \theta^* \pmod{2\pi}, \ \mu = \mu(\varepsilon) \tag{7}$$

Let us assume that

$$\det \frac{\partial f}{\partial \bar{x}}(0, 0, \theta^*, 0) = 1 + \det A - \cos\theta^*(a_{11} + a_{22}) - \sin\theta^*(a_{12} - a_{21}) \neq 0 \tag{8}$$

Then the first equation of (7) will be single valued for sufficiently small $|\bar{x}|, |\varepsilon|, |\mu|$, and analytically solvable in $\bar{x}$: $\bar{x} = \bar{x}(\varepsilon, \mu) = O(|\varepsilon| + |\mu|)$. Choosing a sequence $\{\varepsilon_n\} \to 0$ satisfying the second equation of the system (7), $\mu = \mu(\varepsilon_n)$, we obtain a sequence of solutions of the system (7) $\{\bar{x}_n = \bar{x}(\varepsilon_n, \mu(\varepsilon_n))\} \to 0$. From this we find that the sequence $\{h_n = \varkappa \varepsilon_n^2\} \to 0$ has a corresponding sequence of solutions of (5) $\{x_n = x(h_n) = \varepsilon_n \bar{x}_n\} \to 0$, QED.

Thus it remains to show that θ^* satisfying the condition (8) exists, This can be shown as follows. The canonical character of the system (1) implies that the mapping $(\varphi_h^1)^{-1}$ preserves the oriented area $\int d\xi_2 \wedge d\eta_2$, therefore $\det A = 1$ and the inequality (8) has a solution in θ^*, which completes the proof of the theorem.

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